

# THE MULTIPLE HOLOMORPH OF A FINITELY GENERATED ABELIAN GROUP

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ABSTRACT. W.H. Mills has determined, for a finitely generated abelian group  $G$ , the regular subgroups  $N \cong G$  of  $S(G)$ , the group of permutations on the set  $G$ , which have the same holomorph of  $G$ , that is, such that  $N_{S(G)}(N) = N_{S(G)}(\rho(G))$ , where  $\rho$  is the (right) regular representation.

We give an alternative approach to Mills' result, which relies on a characterization of the regular subgroups of  $N_{S(G)}(\rho(G))$  in terms of commutative ring structures on  $G$ .

We are led to solve, for the case of a finitely generated abelian group  $G$ , the following problem: given an abelian group  $(G, +)$ , what are the commutative ring structures  $(G, +, \cdot)$  such that all automorphism of  $G$  as a group are also automorphisms of  $G$  as a ring?

## 1. INTRODUCTION

Let  $G$  be a group, and  $\rho : G \rightarrow S(G)$  its right regular representation, where  $S(G)$  is the group of permutations on the set  $G$ . The normalizer

$$\text{Hol}(G) = N_{S(G)}(\rho(G))$$

of the image of  $\rho$  is the *holomorph* of  $G$ , and it is isomorphic to the natural extension of  $G$  by its automorphism group  $\text{Aut}(G)$ . The *multiple holomorph* of  $G$  has been defined in G.A. Miller [Mil08] as

$$N_{S(G)}(\text{Hol}(G)) = N_{S(G)}(N_{S(G)}(\rho(G))).$$

Miller has shown that the quotient group

$$T(G) = N_{S(G)}(\text{Hol}(G)) / \text{Hol}(G)$$

acts regularly by conjugation on the set of the regular subgroups  $N$  of  $S(G)$  which are isomorphic to  $G$  and have the same holomorph of  $G$ , that is, the regular subgroups  $N \cong G$  of  $S(G)$  such that

$$N_{S(G)}(N) = N_{S(G)}(\rho(G)).$$

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There has been some attention in the recent literature [Koh15] to the problem of determining, for  $G$  in a given class of groups, the set

$$\mathcal{H}(G) = \left\{ N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \text{Hol}(G) \right\}$$

and the group  $T(G)$ .

In 1951 W.H. Mills [Mil51] determined these data for a finitely-generated abelian group  $G$ . In this paper, we redo Mills' work using the approach of [CDVS06], which allows us to translate the problem in terms of commutative rings. In particular, we are led to solve the following question, which might be of independent interest, for the case when  $G$  is a finitely-generated abelian group.

**Question.** *Let  $(G, +)$  be an abelian group.*

*What are the commutative ring structures  $(G, +, \cdot)$  such that the automorphisms of  $G$  as a group are also automorphisms of  $G$  as a ring?*

Theorem 5.2 states that if  $G$  is a finitely-generated abelian group, then  $T(G)$  is an elementary abelian 2-group, of order 1, 2, or 4. In other words, for a given  $G$ , there are either 1, 2, or 4 regular subgroups  $N$  of  $S(G)$  that are isomorphic to  $G$ , and such that  $N_{S(G)}(N) = N_{S(G)}(\rho(G))$ .

These regular subgroups are described, via the just mentioned ring connection, in Theorem 4.12. We also determine explicitly all the group structures  $G$  for which  $|T(G)| > 1$ .

The plan of the paper is the following. In Section 2 we define the various holomorphs, and set up the problem. In Section 3 we rephrase the problem in terms of rings. The classification of the rings is worked out in Section 4. The group  $T(G)$  is discussed in Section 5.

## 2. GROUPS WITH THE SAME HOLOMORPH

In this section,  $G$  is a finitely-generated abelian group, written additively.

The *holomorph* of a group  $G$  is the natural semidirect product

$$\text{Aut}(G) G$$

of  $G$  by its automorphism group  $\text{Aut}(G)$ . Let  $S(G)$  be the group of permutations on the set  $G$ . Consider the (right) regular representation

$$\begin{aligned} \rho : G &\rightarrow S(G) \\ g &\mapsto (x \mapsto x + g). \end{aligned}$$

The following is well-known.

**Proposition 2.1.**  *$N_{S(G)}(\rho(G)) = \text{Aut}(G) \rho(G)$  is isomorphic to the holomorph  $\text{Aut}(G) G$  of  $G$ .*

**Definition 2.2.** *We write  $\text{Hol}(G) = N_{S(G)}(\rho(G))$ . We will refer to either of the isomorphic groups  $N_{S(G)}(\rho(G))$  and  $\text{Aut}(G) G$  as the holomorph of  $G$ .*

One may inquire, what are the regular subgroups  $N \leq S(G)$  which have the same holomorph as  $G$ , that is, for which

$$(2.1) \quad \text{Hol}(N) \cong N_{S(G)}(N) = N_{S(G)}(\rho(G)) = \text{Hol}(G).$$

W.H. Mills has noted in [Mil51] that if (2.1) holds, then  $G$  and  $N$  need not be isomorphic.

When we restrict our attention to the regular subgroups  $N$  of  $S(G)$  for which  $N_{S(G)}(N) = \text{Hol}(G)$  and  $N \cong G$ , we can appeal to a result of G.A. Miller [Mil08]. Miller found a characterization of these subgroups in terms of the *multiple holomorph* of  $G$

$$N_{S(G)}(\text{Hol}(G)) = N_{S(G)}(N_{S(G)}(\rho(G))).$$

Consider the set

$$\mathcal{H}(G) = \{N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \text{Hol}(G)\}.$$

Using the well-known fact that two regular subgroups of  $S(G)$  are isomorphic if and only if they are conjugate in  $S(G)$ , Miller showed that the group  $N_{S(G)}(\text{Hol}(G))$  acts transitively on  $\mathcal{H}(G)$  by conjugation. Clearly the stabilizer in  $N_{S(G)}(\text{Hol}(G))$  of any element  $N \in \mathcal{H}(G)$  is  $N_{S(G)}(N) = \text{Hol}(G)$ . We obtain

**Theorem 2.3.** *The group*

$$T(G) = N_{S(G)}(\text{Hol}(G)) / \text{Hol}(G)$$

*acts regularly on  $\mathcal{H}(G)$  by conjugation.*

### 3. REGULAR NORMAL SUBGROUPS OF THE HOLOMORPH

Given an abelian group  $G$ , we aim first at giving a description of the set

$$\mathcal{K}(G) = \{N \leq S(G) : N \text{ is regular, } N \trianglelefteq \text{Hol}(G)\} \supseteq \mathcal{H}(G).$$

It was noted in [FCC12] that the results of [CDVS06] on affine groups admit a straightforward extension to the case of holomorphs of abelian groups. We recall this here in our context.

Let  $N \leq \text{Hol}(G)$  a regular subgroup. Write  $\nu(g)$ , with  $g \in G$ , for the unique element of  $N$  such that  $0^{\nu(g)} = g$ . (We write group actions as exponents.) Then there is a map  $\gamma : G \rightarrow \text{Aut}(G)$  such that for  $g \in G$  we can write uniquely

$$(3.1) \quad \nu(g) = \gamma(g)\rho(g).$$

For  $g, h \in G$  we have

$$(3.2) \quad \nu(g)\nu(h) = \gamma(g)\rho(g)\gamma(h)\rho(h) = \gamma(g)\gamma(h)\rho(g^{\gamma(h)} + h).$$

Since  $N$  is a subgroup of  $S(G)$ , and the expression (3.1) is unique, we obtain, for  $g, h \in G$ ,

$$(3.3) \quad \gamma(g)\gamma(h) = \gamma(g^{\gamma(h)} + h).$$

Note, for later usage, that (3.3) can be rephrased, setting  $k = g^{\gamma(h)}$ , as

$$(3.4) \quad \gamma(k + h) = \gamma(k^{\gamma(h)^{-1}})\gamma(h),$$

for  $h, k \in G$ .

To enforce  $N \trianglelefteq \text{Hol}(G)$ , it is now enough to make sure that  $N$  is normalized by  $\text{Aut}(G)$ . In fact,  $N$  is a transitive subgroup of  $\text{Hol}(G)$ , so that  $\text{Hol}(G)$  is the product of  $N$  and the stabilizer of 0 in  $\text{Hol}(G)$ , which is  $\text{Aut}(G)$ .

In order for  $\text{Aut}(G)$  to normalize  $N$ , we must have that for all  $\beta \in \text{Aut}(G)$  and  $g \in G$ , the conjugate  $\nu(g)^\beta$  of  $\nu(g)$  by  $\beta$  in  $S(G)$  lies in  $N$ . Since

$$\nu(g)^\beta = (\gamma(g)\rho(g))^\beta = \gamma(g)^\beta \rho(g)^\beta = \gamma(g)^\beta \rho(g^\beta),$$

uniqueness of (3.1) implies that this is equivalent to

$$(3.5) \quad \gamma(g^\beta) = \gamma(g)^\beta$$

for  $g \in G$  and  $\beta \in \text{Aut}(G)$ . Applying this to (3.4), we obtain

$$(3.6) \quad \gamma(k + h) = \gamma(k^{\gamma(h)^{-1}})\gamma(h) = \gamma(k)^{\gamma(h)^{-1}}\gamma(h) = \gamma(h)\gamma(k),$$

that is,  $\gamma : G \rightarrow \text{Aut}(G)$  is a homomorphism, as  $G$  is abelian.

Note that (3.3) follows from (3.5) and (3.6), as

$$\gamma(g^{\gamma(h)} + h) = \gamma(g)^{\gamma(h)}\gamma(h) = \gamma(g)\gamma(h).$$

We now state the characterization we will be exploiting in the rest of the paper.

**Theorem 3.1.** *Let  $G$  be an abelian group. The following data are equivalent.*

- (1) *An abelian regular subgroup  $N \trianglelefteq \text{Hol}(G)$ , that is, an element of  $\mathcal{K}(G)$ .*
- (2) *A homomorphism*

$$\gamma : G \rightarrow \text{Aut}(G)$$

*such that for  $g \in G$  and  $\beta \in \text{Aut}(G)$*

$$(3.7) \quad \gamma(g^\beta) = \gamma(g)^\beta.$$

- (3) *A commutative rings structure  $(G, +, \cdot)$  such that*
  - (a) *the operation  $g \circ h = g + h + gh$  defines a group structure  $(G, \circ)$ ,*
  - (b)  *$ghk = 0$  for all  $g, h, k \in G$ , and*
  - (c) *each automorphism of the group  $(G, +)$  is also an automorphism of the ring  $(G, +, \cdot)$ .*

Moreover, under these assumptions

- (i) *in terms of (2), the operations of (3) are given by*

$$g \cdot h = -g + g^{\gamma(h)}, \quad \text{and} \quad g \circ h = g^{\gamma(h)} + h.$$

*for  $g, h \in G$ .*

- (ii) The function  $\nu$  of (3.2) defines an isomorphism  $(G, \circ) \rightarrow N$ .
- (iii) Every automorphism of  $G$  is also an automorphism of  $(G, \circ)$ .

Note that (3b) implies (3a).

*Proof.* We have already seen that (1) and (2) are equivalent.

We now recall from [CDVS06, FCC12] that if  $N$  is a regular abelian subgroup of  $\text{Hol}(G)$ , and  $\gamma$  is the associated function as in (2), then, setting, for  $g, h \in G$

$$g \cdot h = -g + g^{\gamma(h)},$$

we obtain a ring structure  $(G, +, \cdot)$  on  $G$  such that

$$g \circ h = g + h + gh = g^{\gamma(h)} + h$$

defines a group structure  $(G, \circ)$ .

To show that (2) implies (3b), we have to prove that for all  $h, k \in G$  we have  $\gamma(hk) = 1$ . (This was already observed in a comment after Lemma 3 of [CDVS06].) In fact

$$\gamma(hk) = \gamma(h^{\gamma(k)} - h) = \gamma(h)^{\gamma(k)} \gamma(h)^{-1} = [\gamma(k), \gamma(-h)] = 1,$$

as  $\gamma : G \rightarrow \text{Aut}(G)$  is a homomorphism, and  $G$  is abelian.

To show that (2) implies (3c), let  $h, k \in G$ , and  $\beta \in \text{Aut}(G)$ . We have

$$\begin{aligned} h^\beta \cdot k^\beta &= -h^\beta + h^{\beta\gamma(k^\beta)} = -h^\beta + h^{\beta\gamma(k)^\beta} = \\ &= -h^\beta + h^{\gamma(k)^\beta} = (-h + h^{\gamma(k)})^\beta = (h \cdot k)^\beta, \end{aligned}$$

where we have used (3.7).

The bijection  $\nu$  introduced above is a homomorphism  $(G, \circ) \rightarrow N$  by (3.2) and (3.3).

Finally, (iii) follows from

$$(g \circ h)^\beta = (g + h + gh)^\beta = g^\beta + h^\beta + g^\beta h^\beta = g^\beta \circ h^\beta$$

for  $g, h \in G$  and  $\beta \in \text{Aut}(G)$ .

Conversely, given a ring as in (3), the following calculations show that the function  $\gamma : G \rightarrow S(G)$  given by  $\gamma(g) : h \mapsto h + hg$  satisfies the conditions of (2). (Here  $\gamma(g) \in S(G)$  because  $\gamma(g)\gamma(-g) : h \mapsto (h + hg) + (h + hg)(-g) = h + h(g - g) = h$ , where we have used (3b).)

$$(h + k)^{\gamma(g)} = h + k + (h + k)g = h + hg + k + kg = h^{\gamma(g)} + k^{\gamma(h)},$$

for all  $g, h, k \in G$ , shows that  $\gamma$  maps  $G$  into  $\text{Aut}(G)$ .

$$g^{\gamma(h)\gamma(k)} = (g + gh) + (g + gh)k = g + g(h + k) = g^{\gamma(h+k)},$$

for all  $g, h, k \in G$ , where we have used (3b), shows that  $\gamma : G \rightarrow \text{Aut}(G)$  is a morphism.

$$h^{\gamma(g^\beta)} = h + hg^\beta = (h^{\beta^{-1}} + h^{\beta^{-1}}g)^\beta = h^{\beta^{-1}\gamma(g)\beta} = h^{\gamma(g)^\beta},$$

for all  $g, h \in G$ , and  $\beta \in \text{Aut}(G)$ , shows that  $\gamma$  satisfies (3.7).  $\square$

Suppose the finitely generated abelian group  $(G, +)$  admits a ring structure  $(G, +, \cdot)$  as in Theorem 3.1.(3). Taking  $\beta \in \text{Aut}(G, +)$  to be inversion  $g \mapsto -g$ , we get that for all  $g, h \in G$  one has

$$-gh = (-g)(-h) = gh,$$

that is, all products satisfy

$$(3.8) \quad 2 \cdot gh = 0.$$

We have obtained

**Lemma 3.2.** *In the commutative ring  $(G, +, \cdot)$  as in Theorem 3.1.(3) we have*

- (1)  $2 \cdot gh = 0$ , for all  $g, h \in G$ , so that
- (2) if  $(G, +)$  has no elements of order 2, ring multiplication is trivial, and
- (3)  $(g + h)^2 = g^2 + h^2$ , for all  $g, h \in G$ .

#### 4. THE CLASSIFICATION

From now on, let  $G$  be a finitely generated abelian group.

For such  $G$ , Mills [Mil51] has determined the set

$$\mathcal{H}(G) = \{N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \text{Hol}(G)\}.$$

In the following we will first determine the set

$$\mathcal{K}(G) = \{N \leq S(G) : N \text{ is regular, } N \trianglelefteq \text{Hol}(G)\} \supseteq \mathcal{H}(G),$$

weeding out in the process the  $N$  which are not isomorphic to  $G$ . In Section 5 we will show that the remaining groups are precisely the elements of  $\mathcal{H}(G)$ , and we will also determine the group  $T(G)$ .

According to Theorem 3.1, we proceed to find all ring structures  $(G, +, \cdot)$  such that all automorphisms of  $G$  as a group are also automorphisms of  $G$  as a ring. We will usually tacitly ignore the trivial case when  $G^2 = \{xy : x, y \in G\} = \{0\}$ .

If  $a \in G$  has odd order  $d$ , then according to Lemma 3.2(1), for all  $b \in G$  we have  $ab = d(ab) = (da)b = 0$ . Therefore the odd part of  $G$  is in the annihilator. We will from now on assume

$$G = F \times H,$$

where  $F$  is free abelian of finite rank, and  $H$ , the torsion part, is a finite 2-group. We write

$$\Omega(H) = \{t \in H : 2t = 0\}.$$

By Lemma 3.2.(1), all products in the ring  $(G, +, \cdot)$  lie in  $\Omega(H)$ . We regard  $\Omega(H)$  as a vector space over the field  $\mathbf{E} = \{0, 1\}$  with 2 elements.

#### 4.1. The case $F = 0$ .

We first discuss the structure of the 2-torsion part  $H$  in the case when the torsion-free part  $F$  is zero.

Write

$$H = \prod_{i=1}^m \langle x_i \rangle,$$

where  $|x_i| = 2^{e_i}$ , with  $e_i > 0$ , and  $|x_i| \geq |x_j|$  for  $i \geq j$ . Write  $t_i = 2^{e_i-1}x_i$  for the involution in  $\langle x_i \rangle$ .

A *homogeneous component* of  $H$  will be a subgroup  $\prod_{i=a}^b \langle x_i \rangle$ , for some  $a \leq b$ , such that  $|x_{a-1}| > |x_a| = |x_{a+1}| = \cdots = |x_b| > |x_{b+1}|$ , where the first inequality does not occur if  $a = 1$ , and the last one does not occur if  $b = m$ .

Consider the following automorphisms of  $H$ .

- (1)  $\xi_{ij}$ , for  $x_i, x_j$  in the same homogeneous component, exchanges  $x_i$  with  $x_j$ , and leaves all the other  $x_k$  fixed.
- (2)  $\gamma_{ij}$ , for  $i < j$ , maps  $x_i$  to  $x_i + x_j$ , and leaves all the other  $x_k$  fixed.
- (3)  $\beta_{ij}$ , for  $i > j$ , maps  $x_i$  to  $x_i + 2^{e_j-e_i}x_j$ , and leaves all the other  $x_k$  fixed. Note that  $\beta_{ij}$  maps  $t_i$  to  $t_i + t_j$ .

##### 4.1.1. Torsion case, $m = 1$ .

If ring multiplication is non-trivial, then  $x_1^2 = t_1 \neq 0$ .

If  $|x_1| = 2$ , we obtain  $t_1^2 = t_1$ , and thus  $t_1^3 = t_1$ , contradicting Theorem 3.1(3b).

If  $|x_1| = 4$ , we obtain  $x_1 \circ x_1 = 2x_1 + t_1 = 0$ , so  $(H, \circ)$  is elementary abelian, and therefore not isomorphic to  $H$ .

If  $|x_1| > 4$ , we obtain a group  $(H, \circ)$  isomorphic to  $H$ , as it is clearly cyclic, generated by  $x_1$ . This yields the two rings

$$(4.1) \quad \begin{cases} n = 0, m = 1 \\ |x_1| > 4 \\ x_1^2 \in \{0, t_1\} \end{cases}$$

**Fact 4.1.** *If  $m \geq 2$ , then*

$$x_1x_2 = \eta_1t_1 + \eta_2t_2 \neq t_2, \quad \text{for some } \eta_i \in \mathbf{E}.$$

*Proof.* We have  $x_1x_2 = \sum_{i=1}^m \eta_i t_i$  for some  $\eta_i \in \mathbf{E}$ . Applying  $\beta_{j1}$  to this, for  $j > 2$ , we see that

$$x_1x_2 = (x_1x_2)\beta_{j1} = \left(\sum_{i=1}^m \eta_i t_i\right)\beta_{j1} = \left(\sum_{i=1}^m \eta_i t_i\right) + \eta_j t_1,$$

whence  $\eta_j = 0$  for  $j > 2$ .

It remains to show that  $x_1x_2 \neq t_2$ . If  $x_1x_2 = t_2$ , in the case when  $|x_1| = |x_2|$  we have  $x_1x_2 = (x_1x_2)\xi_{12} = t_1$ , a contradiction; when  $|x_1| > |x_2|$ , that is,  $e_1 > e_2$  we have  $t_1 + t_2 = t_2\beta_{21} = (x_1x_2)\beta_{21} =$

$x_1(x_2 + 2^{e_1 - e_2}x_1) = x_1x_2$ , a contradiction, using the fact that Lemma 3.2 implies  $2^{e_1 - e_2}x_1^2 = 0$ .  $\square$

**Fact 4.2.** *Suppose  $m > 2$ ,  $|x_1| > |x_3|$ , and*

- *either  $|x_2| > |x_3|$ ,*
- *or  $\eta_2 = 0$  in Fact 4.1.*

*Then  $x_1x_j = x_2x_i = x_ix_j = 0$  for  $i, j > 2$ .*

*Proof.* Let  $i, j > 2$ .

Note that  $|x_1| > |x_3| \geq |x_i|$  implies

$$t_1\gamma_{1i} = (2^{e_1-1}x_1)\gamma_{1i} = 2^{e_1-1}(x_1 + x_i) = 2^{e_1-1}x_1 = t_1.$$

Similarly, if  $|x_2| > |x_3|$  we have  $t_2\gamma_{2j} = t_2$ . Thus under the given hypotheses we have, for  $\eta_1, \eta_2 \in \mathbf{E}$ ,

$$(\eta_1t_1 + \eta_2t_2)\gamma_{1i} = \eta_1t_1 + \eta_2t_2 = (\eta_1t_1 + \eta_2t_2)\gamma_{2j}.$$

Apply  $\gamma_{1i}$  to  $x_1x_2$ , and use Fact 4.1, to get

$$x_1x_2 = (x_1x_2)\gamma_{1i} = (x_1 + x_i)x_2 = x_1x_2 + x_ix_2,$$

whence  $x_ix_2 = 0$ .

Apply  $\gamma_{2j}$  to  $x_1x_2$  to get

$$x_1x_2 = (x_1x_2)\gamma_{2j} = x_1(x_2 + x_j) = x_1x_2 + x_1x_j,$$

whence  $x_1x_j = 0$ .

Finally, apply  $\gamma_{1i}\gamma_{2j}$  to  $x_1x_2$  to get

$$x_1x_2 = (x_1x_2)\gamma_{1i}\gamma_{2j} = (x_1 + x_i)(x_2 + x_j) = x_1x_2 + x_ix_j,$$

whence  $x_ix_j = 0$ .  $\square$

#### 4.1.2. Torsion case, $m \geq 2$ , $x_1x_2 = t_1$ .

If  $|x_1| = |x_2|$ , applying  $\xi_{12}$  to  $x_1x_2 = t_1$  we get  $x_1x_2 = t_2$ , a contradiction.

If  $|x_1| > |x_2|$ , we have  $2^{e_1-1}x_2 = 0$ , so that  $t_1 = 2^{e_1-1}x_1 = 2^{e_1-1}(x_1 + x_2) = 2^{e_1-1}(x_1\gamma_{12}) = t_1\gamma_{12}$ , which implies  $x_1x_2 = t_1 = t_1\gamma_{12} = (x_1x_2)\gamma_{12} = (x_1 + x_2)x_2 = x_1x_2 + x_2^2$ , so that  $x_2^2 = 0$ . Applying  $\gamma_{2i}$  to the last identity, we obtain  $x_i^2 = 0$  for  $i > 2$ .

Using Fact 4.2 we obtain  $x_1x_j = x_2x_i = x_ix_j = 0$  for  $i, j > 2$ .

If  $m > 2$  and  $|x_2| = |x_3|$ , we have  $t_1 = t_1\xi_{23} = (x_1x_2)\xi_{23} = x_1x_3$ , a contradiction.



We have obtained the following rings sharing the same group structure.

$$(4.2) \quad \begin{cases} n = 0, m \geq 2 \\ |x_1| > |x_2|, \\ |x_1| > 4 & \text{if } x_1^2 \neq 0, \\ |x_2| > |x_3| & \text{if } m > 2 \text{ and } x_1x_2 \neq 0 \\ x_1^2 \in \{0, t_1\} \\ x_i^2 = 0, & \text{for } i > 1, \\ x_1x_2 \in \{0, t_1\}, \\ x_1x_i = x_2x_j = x_ix_j = 0, & \text{for } i, j > 2. \end{cases}$$

These are two rings if  $m > 2$  and  $|x_2| = |x_3|$ , four rings otherwise.

Here we have required  $|x_1| > 4$ , as in 4.1.1, to make sure that  $x_1$  retains its order in  $(H, \circ)$ .

**Remark 4.3.** Clearly the elements  $x_i$  retain their orders in  $(H, \circ)$ . Moreover it is easy to see that  $(H, \circ)$  is still the direct product of the subgroups spanned by the  $x_i$ , so that  $(H, \circ)$  is isomorphic to  $H$ .

**Remark 4.4.** In (4.2), the first lines before the occurrence of products in the ring define the group structure. The same applies in the rest of this section.

4.1.3. *Torsion case,  $m \geq 2$ ,  $x_1x_2 = t_1 + t_2$ .*

If  $|x_1| > |x_2|$ , then  $t_1 + t_2 = x_1x_2 = (x_1x_2)\beta_{21} = (t_1 + t_2)\beta_{21} = t_2$ , a contradiction.

Therefore  $|x_1| = |x_2|$ . Applying  $\gamma_{12}$  and  $\beta_{12}$  to  $x_1x_2 = t_1 + t_2$  we obtain  $x_1^2 = t_1$  and  $x_2^2 = t_2$ .

When  $m > 2$ , if  $|x_2| = |x_3|$ , applying  $\xi_{23}$  to  $x_1x_2 = t_1 + t_2$  we obtain  $x_1x_3 = t_1 + t_3$ . But then applying  $\gamma_{23}$  to  $x_1x_2 = t_1 + t_2$  we obtain  $t_2 + t_3 = x_1x_2 + x_1x_3 = t_1 + t_2 + t_3$ , a contradiction.

Therefore if  $m > 2$  we have  $|x_2| > |x_3|$ . Using Fact 4.2 we obtain  $x_1x_j = x_2x_i = x_ix_j = 0$  for  $i, j > 2$ . We have obtained the ring with non-trivial multiplication

$$(4.3) \quad \begin{cases} n = 0, m \geq 2 \\ |x_1| = |x_2| > 4 \\ |x_2| > |x_3| \text{ if } m > 2 \\ x_1^2 = t_1, x_2^2 = t_2, \\ x_1x_2 = t_1 + t_2, \\ x_1x_i = x_2x_j = x_ix_j = 0, & \text{for } i, j > 2. \end{cases}$$

The same group obviously allows also trivial ring multiplication. Note that we have to take  $|x_1| > 4$  here, for the same argument of 4.1.1. And  $(H, \circ)$  is isomorphic to  $H$ , as per Remark 4.3.

4.1.4. *Torsion case,  $m \geq 2$ ,  $x_1x_2 = 0$ .*

Applying  $\gamma_{12}$  to  $x_1x_2 = 0$  we obtain  $x_2^2 = 0$ .

The arguments of the proof of Fact 4.2 yield  $x_1x_j = x_2x_i = x_ix_j = 0$  for  $i, j > 2$ .

If  $|x_1| = |x_2|$ , then applying  $\xi_{12}$  to  $x_2^2 = 0$  we obtain  $x_1^2 = 0$ , and the ring has trivial multiplication.

Therefore  $|x_1| > |x_2|$ , and the ring is described in (4.2).

#### 4.2. The case $F \neq 0$ .

Write

$$F = \prod_{i=1}^n \langle z_i \rangle,$$

with all  $z_i \neq 0$ .

Consider the following automorphisms of  $F \times H$ , which are trivial on  $H$ .

- (1)  $\Xi_{ij}$ , for  $i \neq j$  exchanges  $z_i$  with  $z_j$ , and leaves  $H$  and all the other  $z_k$  fixed.
- (2)  $\Gamma_{ij}$ , for  $i \neq j$ , maps  $z_i$  to  $z_i + z_j$ , and leaves  $H$  and all the other  $z_k$  fixed.
- (3)  $\zeta_{ig}$ , for  $g \in H$ , maps  $z_i$  to  $z_i + g$ , and leaves  $H$  and all the other  $z_k$  fixed.

Recall that  $\Omega(H) = \{t \in G : 2t = 0\}$ . By Lemma 3.2, the ring product on  $G$  yields a bilinear map

$$G/2G \times G/2G \rightarrow \Omega(H).$$

Since the automorphisms  $X_{ij}, \Gamma_{ij}, \zeta_{ig}, \xi_{ij}, \gamma_{ij}, \beta_{ij}$  generate all the automorphisms of  $F/2F \times H/2H$ , it will be easy to see that all rings  $(G, +, \cdot)$  constructed in the following have the property that all the automorphisms of the group  $(G, +)$  are also automorphisms of the ring.

Since by Lemma 3.2 the square map  $z \rightarrow z^2$  is a group homomorphism  $F \rightarrow \Omega(H)$ , we may make the following

**Assumption 4.5.** *The indexing of the  $z_i$  is chosen so that if some square of the  $z_i$  is non-zero, then  $z_1^2 \neq 0$ .*

We first record the following well-know fact, which in our context can be seen using the  $\beta_{i1}$ .

**Lemma 4.6.** *Let  $P \neq 1$  be a finite, abelian  $p$ -group. The following are equivalent:*

- (1)  $P$  has a characteristic minimal subgroup, that is, a characteristic subgroup of order  $p$ ,
- (2)  $P$  has a unique characteristic minimal subgroup, and
- (3)  $P$  is the direct product of a cyclic group of order  $p^e$ , for some  $e \geq 1$ , by a group of exponent less than  $p^e$ .

*If these conditions are verified, the unique characteristic subgroup of order  $p$  is  $P^{p^{e-1}}$ .*

Write

$$F^2 = \{ab : a, b \in F\} \subseteq \Omega(H)$$

for the set of products of elements of  $F$ .

**Fact 4.7.**

- (1)  $\text{Aut}(H)$  acts trivially on the set  $F^2$ .
- (2) The set  $F^2 \subseteq \Omega(H)$  is either zero or  $\langle t_1 \rangle$ . If it is non-zero, then it is the unique minimal characteristic subgroup of  $H$ .
- (3) The set  $\{z^2 : z \in F\} \subseteq \Omega(H)$  is either zero or  $\langle t_1 \rangle$ . If it is non-zero, then it is the unique minimal characteristic subgroup of  $H$ , and  $n = 1$ .

*Proof.* To see (1), take an arbitrary automorphism of  $H$ , and extend it trivially to  $F$ .

Let  $u$  be an arbitrary non-zero element of  $F^2$ . Then  $\langle u \rangle$  is a characteristic minimal subgroup of  $H$ , so that by Lemma 4.6 it is the unique characteristic minimal subgroup of  $H$ . This shows (2).

A similar argument yields the first part of (3). If  $\{z^2 : z \in F\} \neq \{0\}$ , then by Assumption 4.5 we have  $z_1^2 \neq 0$ , and thus  $z_1^2 = t_1$ . If  $n > 1$ , applying  $\Xi_{12}$  we see that  $z_2^2 = t_1$ , but then applying  $\Gamma_{12}$  to  $z_1^2 = t_1$  we obtain

$$t_1 = z_1^2 = (z_1 + z_2)^2 = z_1^2 + z_2^2 = 2t_1 = 0,$$

a contradiction. □

**Fact 4.8.** If  $F \neq 0$ , then  $H^2 = 0$ .

*Proof.* Consider arbitrary  $i, j \leq m$ . Since  $z_1 x_i \in H$ , it is fixed by  $\zeta_{1x_j}$ , so that

$$z_1 x_i = (z_1 x_i) \zeta_{1x_j} = (z_1 + x_j) x_i = z_1 x_i + x_j x_i,$$

and thus  $x_i x_j = 0$ . □

**Fact 4.9.** If  $n > 2$ , then  $F^2 = 0$ .

*Proof.* Let  $i, j, k$  be distinct indices. Applying  $\Gamma_{kj}$  to  $z_i z_k$ , we get

$$z_i z_k = z_i (z_k + z_j) = z_i z_k + z_i z_j,$$

whence  $z_i z_j = 0$  for all  $i, j$ . □

If  $z_1 x_1 = 0$ , then applying the  $\Xi_{1i}$  and the  $\gamma_{1j}$  we see that  $z_i x_j = 0$  for all  $i, j$ , that is,  $FH = 0$ .

Let us first consider the case when  $FH \neq 0$ , so that

$$z_1 x_1 = \sum_{i=1}^n \varepsilon_i t_i \neq 0.$$

Applying  $\beta_{i1}$  to this, for  $i > 1$ , we obtain  $z_1 x_1 = \sum_{i=1}^n \varepsilon_i t_i + \varepsilon_i t_1$ , so that  $\varepsilon_i = 0$ , and thus  $z_1 x_1 = t_1$ . Note that this implies  $0 = z_1^2 x_1 = z_1 t_1$ , so that  $|x_1| \geq 4$ . If  $n > 1$ , applying  $\Xi_{12}$  to  $z_1 x_1 = t_1$  we get  $z_2 x_1 = t_1$ , and applying  $\Gamma_{12}$  we get  $t_1 = t_1 \Gamma_{12} = (z_1 x_1) \Gamma_{12} = (z_1 + z_2) x_1 = t_1 + t_1 = 0$ , a contradiction. Therefore  $n = 1$ . We have obtained

**Fact 4.10.** *If  $FH \neq 0$ , then  $n = 1$  and  $z_1x_1 = t_1$ .*

Applying the  $\xi_{1j}$ , we obtain that if

$$|x_1| = |x_2| = \cdots = |x_k| > |x_{k+1}|$$

(where we might have  $k = m$ , so that the final inequality does not occur), then

$$z_1x_i = t_i, \text{ for } i \leq k, \quad z_1x_i = 0, \text{ for } i > k.$$

If  $k > 1$ , this implies  $F^2 = 0$ , by Fact 4.7 and Lemma 4.6. Also, if  $n > 2$ , then  $F^2 = FH = H^2 = 0$ .

We are now able to discuss the possibilities for the products on  $F$ .

4.2.1.  $F \neq 0$ ,  $F^2 = 0$ .

We have the ring with trivial multiplication, or the ring

$$(4.4) \quad \begin{cases} n = 1 \\ |x_1| = |x_2| = \cdots = |x_k| \geq 4, & \text{for some } k \leq m \\ |x_k| > |x_{k+1}|, & \text{if } k < m \\ z_1^2 = 0 \\ z_1x_i = t_i, & \text{for } i \leq k \\ z_1x_i = 0, & \text{for } i > k \\ x_ix_j = 0, & \text{for all } i, j \end{cases}$$

4.2.2.  $F^2 \neq 0$ ,  $z_1^2 \neq 0$ .

By Fact 4.7(3), we have  $n = 1$  here, and

$$z_1^2 = t_1,$$

with  $\langle t_1 \rangle$  characteristic in  $H$ . We obtain the four rings

$$(4.5) \quad \begin{cases} n = 1 \\ m = 1, \text{ or } |x_1| > |x_2| \text{ if } m > 1 \\ |x_1| \geq 4 & \text{if } z_1x_1 \neq 0 \\ z_1^2 \in \{0, t_1\} \\ z_1x_1 \in \{0, t_1\} \\ x_ix_j = 0 & \text{for all } i, j \end{cases}$$

Note that when  $z_1x_1 = 0$  we may well have  $H = \langle x_1 \rangle$  of order 2 here, with  $x_1 = t_1$ .

**Remark 4.11.** *Note that this case comprises (4.4) when  $k = 1$  in (4.4).*

4.2.3. *Torsion-free case,  $F^2 \neq 0$ ,  $z_1^2 = 0$ .*

By Fact 4.9, we have  $n \leq 2$ .

The case  $n = 1$  does not occur, as it means  $F^2 = 0$  here.

If  $n = 2$ , we have

$$z_1 z_2 = t_1$$

by Fact 4.7(2), so that we get the two rings

$$(4.6) \quad \begin{cases} n = 2 \\ m = 1 \text{ and } |x_1| \geq 4, \text{ or } |x_1| > |x_2| \\ z_1^2 = z_2^2 = 0 \\ z_1 z_2 \in \{0, t_1\} \\ z_i x_j = 0, & \text{for all } i, j \\ x_i x_j = 0, & \text{for all } i, j \end{cases}$$

In all of these cases, it is easy to see that  $H$  is isomorphic to  $(H, \circ)$ , as per Remark 4.3.

We can sum up the results of this section in the following Theorem, which represents our main result.

**Theorem 4.12.** *Let  $(G, +)$  be a finitely generated abelian group,*

$$G = F \times H,$$

where

$$F = \prod_{i=1}^n \langle z_i \rangle$$

is torsion-free, of rank  $n$ ,

$$H = \prod_{i=1}^m \langle x_i \rangle$$

is a 2-group, with  $|x_1| \geq |x_2| \geq \dots \geq |x_m|$ .

The possible ring structures with non-trivial multiplication  $(G, +, \cdot)$  on  $(G, +)$ , such that

(1)  $(G, +) \cong (G, \circ)$ , and

(2) all automorphisms of  $(G, +)$  are also automorphisms of  $(G, +, \cdot)$

are those listed under

$$(4.1), (4.2), (4.3), (4.4), (4.5), (4.6).$$

The groups from the different cases (see Remark 4.4) are pairwise non-isomorphic, except for (4.4) and (4.5), as noted in Remark 4.11.

In the cases (4.2) and (4.5) we have two or four rings (including the ring with trivial multiplication) for the same group structure, in the other cases we have two.

All of these  $G$  can be enlarged to  $G \times D$ , where  $D$  is an abelian group of odd order which lies in the annihilator of the ring.

5. THE GROUP  $T(G)$ 

We first record the following

**Lemma 5.1.** *In the notation of Section 3, suppose  $\vartheta \in S(G)$  is an isomorphism  $\vartheta : G \rightarrow (G, \circ)$ .*

*Then  $\vartheta$  conjugates  $\rho(G)$  to  $N$ .*

*Proof.* For  $g, h \in G$  we have

$$g^{\rho(h)^\vartheta} = g^{\vartheta^{-1}\rho(h)\vartheta} = (g^{\vartheta^{-1}} + h)^\vartheta = g \circ h^\vartheta = g^{\nu(h^\vartheta)},$$

whence  $\rho(h)^\vartheta = \nu(h^\vartheta)$ .  $\square$

In the previous section we have determined, for a given finitely generated abelian group  $G$ , all regular subgroups  $N$  of  $S(G)$  which are normal in  $\text{Hol}(G)$ , that is, the elements of the set

$$\mathcal{K}(G) = \{N \leq S(G) : N \text{ is regular, } N \trianglelefteq \text{Hol}(G)\}.$$

We have weeded out those  $N \in \mathcal{K}(G)$  which were not isomorphic to  $G$ . Now if  $N \in \mathcal{K}(G)$  and  $\vartheta : G \rightarrow (G, \circ) \cong N$  is an isomorphism, by Lemma 5.1 we have

$$(5.1) \quad N_{S(G)}(\rho(G))^\vartheta = N_{S(G)}(\rho(G)^\vartheta) = N_{S(G)}(N) \geq N_{S(G)}(\rho(G)).$$

In the rest of this section we will determine, for each of the regular subgroups  $N \cong G$  of the previous section, an isomorphism  $\vartheta : G \rightarrow (G, \circ)$  of order two. We will then have from (5.1)

$$N_{S(G)}(\rho(G)) = N_{S(G)}(\rho(G))^\vartheta^2 \geq N_{S(G)}(\rho(G))^\vartheta,$$

so that

$$N_{S(G)}(N) = N_{S(G)}(\rho(G)).$$

Therefore the regular subgroups  $N \cong G$  of the previous section will turn out to be exactly the elements of the set

$$\mathcal{H}(G) = \{N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \text{Hol}(G)\}.$$

In the previous section we have shown that for each group structure  $(G, +)$  there are 1, 2, or 4 rings  $(G, +, \cdot)$ . Therefore

**Theorem 5.2.** *For each finitely generated abelian group  $G$ , the group  $T(G)$  is elementary abelian, of order 1, 2, or 4*

We now describe, for each of the regular subgroups  $N \cong G$  of the previous section, an isomorphism  $\vartheta : G \rightarrow (G, \circ)$  of order two. In the previous section we have noted that in all the cases of Theorem 4.12, the generators  $z_i, x_j$  are still generators of  $(G, \circ)$ , they retain their orders in  $(G, \circ)$ , and  $(G, \circ)$  is still a direct product of the cyclic subgroups

generated by the  $z_i, x_j$  (see Remark 4.3). Therefore there is an isomorphism  $\vartheta : G \mapsto (G, \circ)$  such that  $z_i^\vartheta = z_i$  and  $x_j^\vartheta = x_j$  for all  $j$ . This can be extended to the whole of  $G$  via

$$(5.2) \quad (x + y)^\vartheta = x^\vartheta \circ y^\vartheta = x^\vartheta + y^\vartheta + x^\vartheta y^\vartheta$$

for all  $x, y \in G$ .

Define a function

$$\begin{aligned} f : G &\rightarrow G \\ u &\mapsto u^\vartheta - u. \end{aligned}$$

We will be using several times the following simple

**Lemma 5.3.**  $f(G) \subseteq G^2$ .

Recall from Lemma 3.2 that  $2G^2 = 0$ , and from Theorem 3.1(3b) that  $G^2$  lies in the annihilator of the ring.

*Proof.* Proceeding by induction on the length of  $u$  as a sum of the generators  $z_i, x_j$ , we have from (5.2), if  $y$  is one of these generators,

$$\begin{aligned} f(u + y) &= (u + y)^\vartheta - (u + y) \\ &= u^\vartheta - u + y^\vartheta - y + u^\vartheta y^\vartheta \\ &= f(u) + u^\vartheta y^\vartheta \in G^2, \end{aligned}$$

as  $y^\vartheta = y$ , for  $y$  a generator. □

Note that for all  $u, v \in G$  we have

$$\begin{aligned} (u + v)^\vartheta &= u^\vartheta + v^\vartheta + u^\vartheta v^\vartheta \\ &= u + f(u) + v + f(v) + (u + f(u))(v + f(v)) \\ &= u + v + f(u) + f(v) + uv, \end{aligned}$$

so that

$$f(u + v) = f(u) + f(v) + uv.$$

Therefore Lemma 5.3 yields  $f(2u) = 2f(u) + u^2 = u^2$ , so that

$$(5.3) \quad f(4u) = f(2u + 2u) = 2f(2u) + 4u^2 = 0.$$

Therefore

$$\begin{aligned} (5.4) \quad u^{\vartheta^2} &= (u + f(u))^\vartheta \\ &= u + f(u) + f(u + f(u)) \\ &= u + f(u) + f(u) + f(f(u)) + uf(u) \\ &= u + f(f(u)), \end{aligned}$$

by Lemma 5.3.

In the cases of Theorem 4.12 when  $|x_1| > 4$ , we have  $f(G) \subseteq G^2 \leq 4H$ . Now (5.4) and (5.3) yield  $u^{\vartheta^2} = u$ .

In the cases when  $|x_1| = 4$ , we have  $G^2 = \langle t_1, \dots, t_k \rangle$  for some  $k$ , and  $x_i^2 = 0$  for all  $i$ . Thus we have, for  $i \leq k$ ,

$$(5.5) \quad f(t_i) = f(2x_i) = 2f(x_i) + x_i^2 = 0,$$

so that (5.4) implies  $u^{\vartheta^2} = u$ .

Finally, when  $|x_1| = 2$  in (4.5), we have  $f(t_1) = f(x_1) = x_1^\vartheta - x_1 = 0$ . Therefore  $\vartheta$  is in all cases an involution, as claimed.

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